$$\frac{.57735^4\left(\frac{.44654^2}{.86603}-.28940^2\right)}{54\times.0000339822}=-26.6 \text{ feet.}$$

Sum of the first and second terms of (3) gives, as the value of x. 6026.5 feet. This result is less than 1 foot in error.

For still greater accuracy either larger values of m may be taken or more than two terms of (A) may be employed.

It is perhaps needless to point out that the same method of evaluating the integrals for y and t may be quite conveniently employed.

OUTLINE OF A COHERENT COURSE IN COLLEGE ALGEBRA.

By DR. A. C. LUNN.

In teaching what may be considered the standard topics in a course on College Algebra it is often difficult to avoid leaving with the students the impression that they have labored over a set of isolated subjects, hard to master because of lack of interrelations. The following outline sketches the result of an attempt to organize many of these subjects into a course, to be developed so that a certain natural unity should be more or less obvious.

§1. The Fundamental Problems.

The main subject is the study of rational functions of a single real variable, in three aspects, the Formula, the Graph, and the Table, denoted in the following by their initial letters. The motto is that every notion, operation, and problem connected with the theory is to be interpreted so far as possible in all three aspects.

By "formula" is understood an algebraic statement of the form $y=f(x)$, giving directions for finding the value of the dependent variable y in terms of the argument x by a certain set of operations, $f(x)$ denoting for the present purpose "an expression in terms of x". The functions or expressions considered here are then either polynomials of the form $A_0x^n+A_1x^{n-1}\ldots\ldots A_{n-1}x+A_n$, or quotients of such, although opportunity is presented for easy passage to certain irrational forms if desired, and certain of the theorems deduced apply to wide classes of functions.

By "graph" is understood a plot of the curve whose equation is $y=f(x)$ in Cartesian coördinates, where the two units of measure are chosen independently according to dictates of convenience for the two axes (not necessarily rectangular), since for present purposes lengths are compared only when parallel.

By a "table" is meant an array of numerical values of y corresponding to assigned values of x, and where not otherwise specified it is understood that the

table is what might be called normal, with the argument values equally spaced; in this case the table includes the columns of differences of various orders as far as found to be needed for completeness.

The primary problems of the theory are then, for a function known in any one of these three aspects, to find means of giving it expression in the other two. This gives six fundamental problems, of which the order of presentation is to some extent arbitrary. The following order is perhaps as good as any.

Problem TG: given the table, to make the graph. This is solved simply by plotting the points given in the table, the inevitable incompleteness of the table being manifest. Practice here is secured well from tables of physical or astronomical observations or statistics, for which no formula is known to the students.

Problem GT: this implies the measurement of chosen ordinates of a given curve, and the insertion of their numerical values in a tabular scheme.

These two problems need be insisted upon only if the idea of plotting is not already somewhat familiar.

Problem FT. For a polynomial this problem is solved for example by the familiar method of synthetic substitution with detached coefficients, implying for each value of the argument an operation such as the following:

$x)$	A_0	A_1	A_2	A_n
		A_0x	$A_0x^2+A_1x$	$A_0x^n+\ldots\ldots+A_{n-1}x$
	A_0	A_0x+A_1	$A_0x^2+A_1x+A_2$	$A_0x^n+\ldots\ldots+A_n$

Many other modes of computation are suggested by the various shapes into which the formulas are thrown for other purposes.

Problem FG. This problem might of course be solved indirectly by computing a table and plotting from the table, or in short, resolving the step FG into the steps FT and TG. But it is much better to do it by purely geometric processes, for which the operations or transformations given in §2 are useful. The manifold ways of using these will be better illustrated in special cases.

Problem TF. Here again the incompleteness of the table is brought vividly home, since it is obviously possible to make many formulas fit the same table of the ordinary kind. But special kinds of solution will be noticed later; in particular the determination of a polynomial of degree n which takes assigned values for $n+1$ given values of the argument.

Problem GF. An outlook is afforded here on the vast fields of mathematical research. If the graph is known to be that of a polynomial of a certain degree the coefficients can be determined by algebraic procees. But the determination of an analytic formula for a given graph is a problem on which has been expended an important part of the mathematical endeavor of a century.

The six problems named, in direct or modified form, make the primary elements of the course, about which all other material is grouped, either as direct outgrowth, or as auxiliary.

§2. The Geometrical Operations.

In the graphic construction of curves which are the graphs of rational functions it is convenient to make use of the following geometric constructions or operations.

(1) The x-Push or Translation through distance a (positive or negative). The curve is moved bodily parallel to the x-axis without changing form, size, or orientation in the plane. The effect on the formula may be indicated thus:

$$^{x}P_{a}f(x)=f(x-a),$$

since the value of the ordinate after the operation must be read as the value previously corresponding to an x-value less by a than its new value.

(2) The y-Push:

$$^{y}P_{b}f(x)=f(x)+b.$$

(3) The x-Stretch or Elongation, with factor m:

$$^{x}S_{m}f(x)=f\left(\frac{x}{m}\right).$$

(4) The y-Stretch, with factor n:

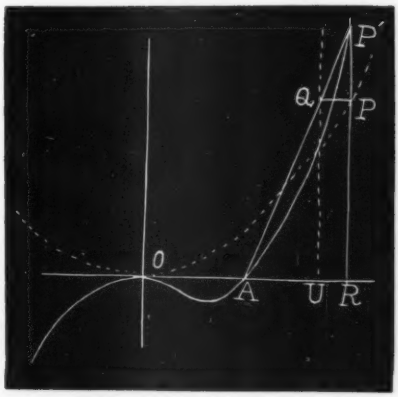
$$^{y}S_{n}f(x)=nf(x).$$

(5) The addition of two curves by algebraic addition of ordinates (operation A).

(6) Multiplication by abscissa difference from assigned point a, where the result is indicated symbolically thus:

$$M_{a}f(x)=(x-a)f(x).$$

This is accomplished by the following construction,* where the original and the final position of the point are marked P and P'.



A is point of abscissa a; U is point of abscissa $a+1$.

$$\frac{y'}{y}=\frac{RP'}{RP}=\frac{RP'}{UQ}=\frac{AR}{AU}=x-a, \quad y'=(x-a)y.$$

*Used by Professor Moore in course in Calculus since 1902; see also Peano, *Applicazioni del Calcolo Infinitesimale*, p. 74.

Here P is given and P' required; the converse is also needed, P' being the known, P the sought point. The effect of this transformation on a given curve is illustrated in the figure, where the original curve (dotted), is $y=x^2$, the new curve (full line), $y=(x-2)x^2$, with the value $a=2$.

These six operations, which will be denoted by the respective symbols indicated, are to be employed in the construction of a curve when the formula is assigned. Illustrations may now be given.

§3. Standard Expressions and Their Graphs.

Start with the line $y=1$, parallel to the x-axis; then the operation M_0 gives the "standard diagonal," whose equation is $y=x$; another application of M_0 gives the parabola $y=x^2$, and by repetition are obtained all the curves of the form $y=x^n$, where n is a positive integer.

With a new start at the line $y=1$, the inverse operation of M_0, which may be called M_0^{-1}, gives the hyperbola $y=x^{-1}$, and by repetition all curves of the type $y=x^{-n}$. By the use of any reference point A on the x-axis instead of the origin may be obtained any curve of the forms $(x-a)^n$, $(x-a)^{-n}$. Or these may be obtained by the operation ${}^xP_a$, applied to the curves x^n, x^{-n}.

A modification of this set is important. Start with the graph of any expression $f(x)$, which does not meet the axis or go to infinity at the point a, and obtain by direct and inverse operations the curves $(x-a)^n f(x)$, $(x-a)^{-n}f(x)$. It is obvious from inspection of the graphs that the general character of the curves in the neighborhood of the point a is the same as for the symple types $(x-a)^n$, $(x-a)^{-n}$, of corresponding exponent. This is expressed by saying that the point a is a 'root' of order n of the function $(x-a)^n f(x)$, and a 'pole' of order n of the function $(x-a)^{-n}f(x)$.

This prepares the way for the systematic study of polynomials of various degrees and their quotients.

The straight line $y=A_0x+A_1$ is constructed from the diagonal $y=x$ by the successive operations ${}^yS_{A_0}$, ${}^yP_{A_1}$, and the interpretation of A_0 as slope and A_1 as y-intercept thus made clear, to yield a shorter method of construction with ruler.

Or with the equation written in the form $y=A_0(x-a)$, where $a=-A_1/A_0$, the same line may be gotten from $y=x$ by x-push of value a, and y-stretch of value A_0.

Or finally the line $y=A_0$ may be operated upon by M, with reference-point $a=-A_1/A_0$.

As the degree increases, varieties of construction multiply rapidly; no attempt at completeness need be made here.

In the case of the parabolas, or quadratic functions, for instance, there are among others the four important forms:

$$y=A_0x^2+A_1x+A_2, \qquad y=A_0(x^2+px+q),$$
$$y=A_0[(x-a)^2+b], \qquad y=A_0(x-r_1)(x-r_2),$$

the final form being considered only when the roots r_1, r_2 are real.

The first form is constructed from the line $y=A_0x+A_1$, by the successive operations M_0, which yields $A_0x^2+A_1x$, and ${}^vP_{A_2}$ which gives the required form. This is plainly the graphic analogue of the synthetic substitution in the formation of the table by computation.

The second form is obtained from the line $y=x+p$ by the operations M_0, ${}^vP_q$, ${}^vS_{A_0}$.

The third form comes from the standard parabola $y=x^2$, by means of ${}^xP_a$, which yields $(x-a)^2$, then ${}^yP_b$ which gives $(x-a)^2+b$, then finally ${}^vS_{A_0}$.

The fourth form results from application of M_{r_2} to the line $A_0(x-r_1)$, or of M_{r_1} to the line $A_0(x-r_2)$.

The character of b as negative, 0, or positive, gives the classification of quadratic expressions as having two, one, or no real roots.

There are here four different choices of the set of three constants needed to determine the parabola, sets (A_0, A_1, A_2), (A_0, p, q), (A_0, a, b), (A_0, r_1, r_2). The determination of the connections between these sets affords a fairly extensive exercise in the manipulation of formulas, the complete problem being to express in terms of the constants of any set those of the remaining three.

For the further illustration of the methods of construction for higher degrees, it will perhaps be sufficient to consider a definite case in some detail.

Let the formula be $y=2x^3-4x^2-22x+24$. Alternative constructions are then:

(I) Operate on line $2x-4$, by M_0, giving $2x^2-4x$; then by ${}^vP_{-22}$, giving $2x^2-4x-22$; then by M_0, giving $2x^3-4x^2-22$; then by ${}^vP_{-22}$, (synthetic substitution).

(II) Write the formula $y=2[(x-\frac{2}{3})^3-\frac{19}{3}x+\frac{33}{27}]$. Here give the standard cubic x^3 an x-push of value $\frac{2}{3}$, add the ordinates of the line $-\frac{19}{3}+\frac{33}{27}$, and make y-stretch with factor 2.

(III) The cubic can be factored to $2(x-4)(x-1)(x+3)$. Operate on line $2(x-4)$ by M_1, then by M_{-3}.

The table of the same cubic may have the form:

x	$f(x)$	Δ_1	Δ_2	Δ_3
3.0	−24.00			
		+ 7.75		
3.5	−16.00		8.50	
		+16.25		1.50
4.0	0.00		10.00	
		+26.25		1.50
4.5	+26.25		11.50	
		+37.75		1.50
5.0	+64.00		13.00	
		+50.75		
5.5	114.75			

§4. Algebraic Theorems.

Sufficient experience in construction of graphs belonging to a variety of formulas leads by induction to the formulation of a number of theorems of which a few may be written down here.

If the graph of a rational function meets the x-axis at the point a, after the manner of the elementary function $(x-a)^n$, its algebraic expression contains the factor $(x-a)^n$ in the numerator; if it goes to infinity like the function $(x-a)^{-n}$ it has the factor $(x-a)^{-n}$ in the denominator. These considerations lead to the general factor-theorem, with the notions of simple and multiple roots and asymptotes, and the attempt to represent the given function by *addition* of elementary functions, each with a single asymptote, developes into a study of resolution into partial fractions.

If the slopes of the secant-lines are obtained by division of the first differences in the table by the x-difference, and their values plotted into a set of curves, one for each chosen value of the x-difference, these curves, "secant-slope curves," are seen to approach coincidence with the tangent-slope curve, or "derived curved," which gives the slope of the tangent line as a function of x. Similarly the rth differences divided by the rth power of the x-difference give curves which approach the rth slope-curve, yielding the successive derivatives of the function. This mode of attack is however, in the case of the higher derivatives, convenient mainly for the polynomials.

Inspection of tables computed from polynomials of various degrees suggests that for a polynomial of degree n the first differences act like a function of degree $n-1$, the second like one of degree $n-2$, and so on, the nth differences being all alike, or a function of degree 0. The proof of this is obtained through the agency of the binomial theorem, proved from the theory of combinations or by induction, permitting the expansion of $(x+w)^n$, $(x+2w)^n$, ..., where w is the value of the x-space of the table, and there may be obtained here directly the formulas for the successive derivatives of x^n, together with the theorems that multiplication by a constant multiplies the derivative, and that the derivative of a polynomial is the sum of the derivatives of the separate terms. Then finally the expansion of any polynomial of degree n is seen to take the form of Taylor's theorem for the expansion of $f(x+h)$, but ending with the nth power of h because of the vanishing of the higher derivatives.

The binomial coefficients ("Pascal Triangle") appear also if an error is made in some entry of the table, this error being propogated into the columns of differences, magnified according to the binomial coefficients with alternating signs.

§5. Interpolation Formula. Linear Equations. Determinants.

Problem TF may be solved in a special case in the form: given $n+1$ entries in a table to find a polynomial of degree n yielding those values. One method of solution is to consider the required function as the sum of n polynomials, each of which goes through one of the given points and the projections

of the remaining n points on the x-axis. This yields immediately the classic interpolation formula of LaGrange.

Another solution is to determine the coefficients directly from the linear equations defining the conditions imposed. This introduces the theory of determinants, which is also suggested by the problem of indeterminate coefficients in the partial fraction expansions, and by the problem of finding the intersection of two linear graphs.

§6. Converse Problem. Determination of Roots.

The previous discussion has considered chiefly the problem, given the value of the argument x, to find the value of the rational function $f(x)$. The converse of this, given the value of $f(x)=b$, to find the value or values of x, is immediately seen to be equivalent to the problem, find the roots of the rational function $f(x)-b$, *i. e.*, the roots of the numerator when the function is written as the quotient of two polynomials. The numerical work is carried out by the Horner method of well-directed trial and error, with change of origin and synthetic substitution.

The preceding sketch is plainly rough and incomplete, but is perhaps sufficient to indicate a trend of thought which has been found to yield abundant material for a quarter's work, without sacrifice of unity of structure. Many other topics may be connected easily with those indicated if time permits.

GRAPHICAL METHODS IN TRIGONOMETRY.

By DR. L. E. DICKSON.

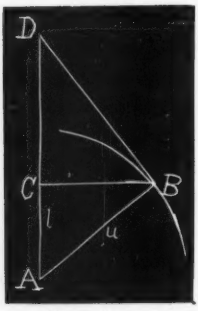
Aside from the important work on the solution of triangles by diagrams drawn to scale, graphic methods are not usually employed in trigonometry. Even if the cartesian graphs of the trigonometric functions are constructed, no serious applications are made of these graphs. They are, however, admirably adapted to the explanation of interpolation, to the visualization and retention in the memory of the ratios for the angles 0°, 90°, etc. (in contrast to their derivation as limiting values), and to the natural solution of trigonometric equations,—in particular, to the visualization of the number of angles $<180°$ having a given sine or cosine. In addition to these minor advantages resulting from a frequent appeal to the graphs, the graphic method may be employed to perform the highly important service of leading the student naturally to the majority of the fundamental trigonometric formulae, including the addition theorem and formulae for conversion of sum into product. This is in marked contrast to the current method by which each formula makes its appearance from some unseen source, to be followed by a more or less artificial proof.

An inspection of the cartesian sine curve reveals two facts (proved by recurring to the unit circle and ordinates used in constructing the graph): the symmetry of each arch and the equality of the various arches. Hence if the sine wave is moved to the right 180° and then reflected on the scale line, it coincides with its former trace as a whole;* hence $\sin(180°+\alpha)=-\sin\alpha$, for *every* angle α. Rotation of the curve about 0° through an angle 180° leads similarly to $\sin(-\alpha)=-\sin\alpha$, for every α. Reflection of the curve on the vertical through the point marked 90° leads to $\sin(180°-\alpha)=\sin\alpha$, for every α. Moving the sine wave 90° to the left, we obtain the cosine wave; hence $\sin(90°+\alpha)=\cos\alpha$, for every α. Performing the last two operations, we get $\sin\alpha=\sin(180°-\alpha)=\cos(90°-\alpha)$. Similarly, all the formulae of this type follow immediately from a combination of three of the preceding operations.

A valuable exercise is afforded by the composition of waves of different periods and phases, with emphasis on the physical applications.

As it seems preferable to *define* the cosecant, secant, and cotangent as the reciprocals of the sine, cosine, and tangent, respectively, it is desirable to construct the graphs of the former direct from the latter.

The following construction for the reciprocal $AD=r$ of a given directed line $AC=l$, the unit of length being u, is very convenient, since it yields r in the position desired for the reciprocal graph. We determine B as an intersection of a circle of center A and radius u with the perpendicular to AC at C. We make angle $ABD=90°$. Then $l:u=u:AD$.



Cartesian graphs may be converted mechanically into polar coördinate graphs. Take a rectangle $ABCD$, whose length AB is approximately π times its height BC, and make alternate forward and backward narrowfolds parallel to BC. The rectangle is compressed into a fluted surface, with D near C, and A near B. If we hold A and B together, but allow the end DC to open, we ultimately obtain a *fan*, whose outline is approximately a semi-circle.† If on the original rectangular strip appeared a cartesian sine arch with ends at A and B (the unit not being too large) and a part of the U-shaped cosecant graph, there will appear on the fan a circle and tangent straight line, representing the polar graphs of sine and cosecant.

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*Hence the equation or graph $y=\sin x$ is transformed into itself by

$$T \;:\; x'=x+180°,\; y'=-y;$$

likewise by the rotation through angle 180° about 0°, viz.,

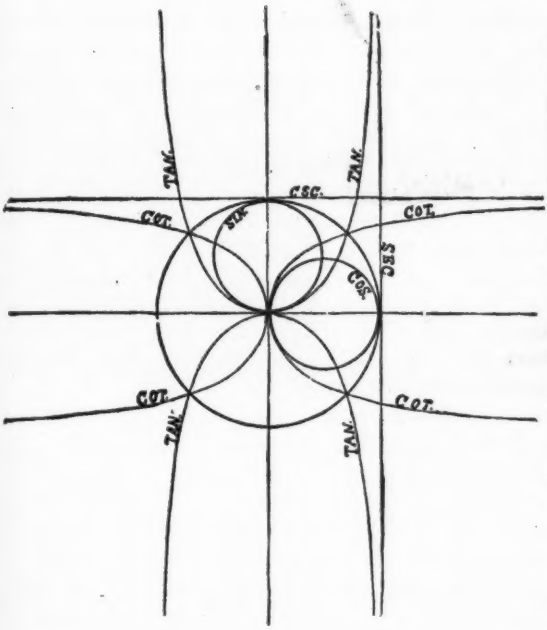
$$S \;:\; x'=-x,\; y'=-y.$$

The product $R=ST$ gives the reflection on the 90° vertical:

$$R \;:\; x'=180°-x,\; y'=y.$$

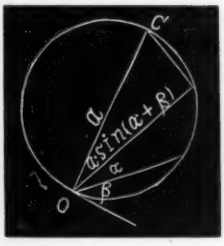
The infinite group transforming $y=\sin x$ into itself is generated by T and S.

†Let the cartesian coordinates of a point P of the rectangle be x, y, the x-axis being AB, the y-axis a perpendicular to AB at its center O. On the fan let the polar coordinates (origin O) of P be r, A. Then $r=y$, while the complement of A is measured by an arc of length x on the semi-circle.



This mechanical derivation of the polar graphs should be followed by their geometric construction on polar coördinate paper, and later by the simple formal proofs that the polar graphs of the sine and cosine are circles with respectively horizontal and vertical tangents at the origin O, and that the graphs of the cosecant and secant are straight lines tangent at the opposite end of the diameter through O. More generally, the polar graph of $r=a\sin(\alpha+\beta)$, where β is a constant angle, is the circle of diameter a, whose tangent at O makes the angle β with the initial line. The graphs of $r=a\sin\alpha$ and $r=a\cos\alpha$ are obtained by setting $\beta=0°$ and $\beta=90°$, respectively.

While the composition of waves on cartesian paper has important physical applications, the composition of the polar graphs offers greater interest as well as greater importance in the mathematical theory. We may give the following definition of composition of graphs: The points (r_1, α), (r_2, α) on the two polar graphs lead to the point (r_1+r_2, α) on the *compound* graph. The following theorem is fundamental:



The compound of a circle on the diameter OP with a circle on the diameter OR is the circle having as diameters the diagonal OQ of the parallelogram $OPQR$.

Let ρ and π be the intersections of an arbitrary line through O with the given circles, and take $\pi S=O\rho$. We are to prove that S lies on the circle with center C at the middle point of PR and radius OC. Draw the perpendicular $C\gamma$ from C to OS. It is to be shown that $O\gamma=S\gamma$. This follows from $\rho\gamma=\pi\gamma$, a consequence of equal intercepts PC and CR between the parallels $P\pi$, $C\gamma$, $R\rho$.

Consider the special case in which $POR=90°$. Set $a=OP$, $b=OR$. Then the equations of the circles on OP, OR, OQ are respectively,

$$r=a\sin\alpha,\quad r=b\cos\alpha,\quad r=\sqrt{(a^2+b^2)}\sin(\alpha+\beta),$$

where $\beta = POQ$ = angle between OR and the tangent at O to the circle on OQ. Hence

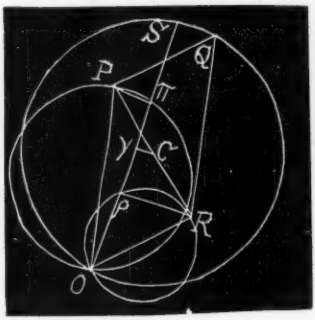
$$a\sin\alpha + b\cos\alpha = \sqrt{(a^2+b^2)}\sin(\alpha+\beta)$$

for every angle α. But

$$a = \sqrt{(a^2+b^2)}\cos\beta,\ b = \sqrt{(a^2+b^2)}\sin\beta.$$

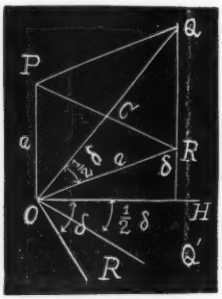
Hence

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$



Consider next the special case $OP = OR = a$. Let OR' and OQ' be the tangents at O to the circles on the diameters OR and OQ. The initial line OH is perpendicular to OP. The circles on OP and OR are $r = a\sin\alpha$, $r = a\sin(\alpha+\delta)$, where $\delta = HOR'$. Since ORQ is isosceles, the exterior angle $\delta = 2ROQ$. Hence $\frac{1}{2}\delta = ROQ = HOQ'$ (the arms of the angles being perpendicular). Hence the compound circle on the diameter OQ is

$$r = (2a\cos\tfrac{1}{2}\delta)\sin(\alpha+\tfrac{1}{2}\delta).$$



Hence

$$\sin\alpha + \sin(\alpha+\delta) = 2\cos\tfrac{1}{2}\delta\sin(\alpha+\tfrac{1}{2}\delta).$$

Since α and δ are arbitrary angles, this formula includes the four formulae expressing $\sin\alpha \pm \sin\gamma$ and $\cos\alpha \pm \cos\gamma$ as products.

Further interesting exercises are obtained from the graphs of $r = b\sin(k\alpha+\delta)$ and their composition with $r = a\sin\alpha$.

As it is desirable to have two essentially different proofs of each theorem, the following alternative proof* of the addition theorem is suggested on account of its simplicity. Let the sum of the given angles α and β be less than 180° (the modifications of the picture for other cases being obvious). It is assumed that the sine of proportion has been previously proved.

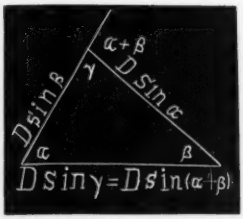
$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = D\ (=\text{diameter of circumscribed circle}).$$

Since the projection of one line on another equals the product of its length by the cosine of the included angle, we obtain from the figure,

$$D\sin(\alpha+\beta) = D\sin\alpha\cos\beta + D\sin\beta\cos\alpha.$$

*Compare Veblen's proof, MONTHLY, January, 1904, p. 7.

Likewise it seems desirable to supplement the usual analytic derivation of $\tan\frac{1}{2}A = r/(s-a)$ by a purely geometric proof (MONTHLY, 1902, p. 36).



Theorem. *The compound of a sphere on the diameter OP with a sphere on the diameter OR is a sphere having as diameter the diagonal OQ of the parallelogram OPQR.*

The proof is similar to the above for circles. If three parallel planes make equal intercepts on one transversal they make equal intercepts on any other transversal.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

228. Proposed by G. W. GREENWOOD, M. A. (Oxon), Professor of Mathematics, McKendree College, Lebanon, Ill.

Sum the infinite series

$$\frac{1}{11.13}+\frac{1}{23.25}+\frac{1}{35.37}+\frac{1}{47.49}+\frac{1}{59.61}+\cdots\cdots \quad \text{[Oxford, 1895].}$$

Solution by the PROPOSER.

We can show that*

$$\frac{1}{2\theta}\left[\frac{1}{\theta}-\cot\theta\right]=\frac{1}{(\pi-\theta)(\pi+\theta)}+\frac{1}{(2\pi-\theta)(2\pi+\theta)}+\frac{1}{(3\pi-\theta)(3\pi+\theta)}+\cdots\cdots$$

Put $\theta=\pi/12$ and we get

$$\frac{\pi}{6}\left[\frac{12-\pi\cot 15^\circ}{\pi}\right]=\frac{144}{\pi^2}\left[\frac{1}{11.13}+\frac{1}{23.25}+\cdots\cdots\right].$$

Hence the required sum $=\dfrac{12-\pi\cot 15^\circ}{24}$.

Also solved by J. Scheffer.

229. Proposed by B. F. YANNEY, Mount Union College, Alliance, O.

If $a_1^n+a_2^n+a_3^n+\cdots\cdots+a_r^n=A^n$, $a_1^m+a_2^m+a_3^m+\cdots\cdots+a_r^m > \text{or} < A^m$, according as $m <$ or $> n$; provided all the letters stand for positive real numbers.

No satisfactory solution has been received.

*Expand $\sin\theta$ in factors, take logarithms of each expression, and differentiate.

230. Proposed by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

Find the value of the determinant of n rows,

$$\begin{vmatrix} 5 & 2 & 0 & 0 & 0 & \cdots \\ 2 & 5 & 2 & 0 & 0 & \cdots \\ 0 & 2 & 5 & 2 & 0 & \cdots \\ 0 & 0 & 2 & 5 & 2 & \cdots \\ 0 & 0 & 0 & 2 & 5 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

[Oxford, 1900.]

Solution by the PROPOSER.

Denote the determinant by u_n. Then $u_n=5u_{n-1}-4u_{n-2}$.
Let $S=u_1+u_2x^2+u_3x^2+\cdots$ Then

$$(1-5x+4x^2)S=u_1+(u_2-5u_1)x=5-4x.$$

$$S=\frac{5-4x}{1-5x+4x^2}=\tfrac{1}{3}\left[\frac{16}{1-4x}-\frac{1}{1-x}\right].$$

Then $u_n=\text{co}\,x^{n-1}$ in $S=\frac{1}{3}[4^{n+1}-1]$.

Also solved by M. E. Graber.

231. Proposed by O. L. CALLECOT, Omaha, Neb.

Sum to infinity: $\dfrac{1}{2.3.4}+\dfrac{1}{5.6.7}+\dfrac{1}{8.9.10}+\cdots$

Remark by Editor EPSTEEN.

Mr. A. H. Holmes notes that

$$\frac{2}{(2+x)(2+x+1)(2+x+2)}=\frac{1}{2+x}-\frac{2}{2+x+1}+\frac{1}{2+x+2}$$

and then substitutes $x=0, 1, 2, \ldots$ From this point on it seems to me that his solution should proceed thus:

$$\frac{1}{1.2.3}+\frac{1}{5.6.7}+\frac{1}{8.9.10}+\cdots$$

$$=\tfrac{1}{2}-\tfrac{2}{3}+\tfrac{1}{4}$$
$$+\tfrac{1}{3}-\tfrac{2}{4}+\tfrac{1}{5}$$
$$+\tfrac{1}{4}-\tfrac{2}{5}+\tfrac{1}{6}$$
$$+\tfrac{1}{5}-\tfrac{2}{6}+\tfrac{1}{7}$$
$$\vdots \qquad \vdots$$

$=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$.

232. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

If one person out of 50 die annually and one person out of 30 is born annually, how long at this rate would be required for the population to treble itself?

Solution by CHRISTIAN HORNUNG, A. M., Heidelberg University, Tiffin, Ohio.

Let a be the population at the beginning; then at the end of the first year the population will be $a+(\frac{1}{30}-\frac{1}{50})a=\frac{76}{75}a$; at the end of the second year $\frac{76}{75}$ of $\frac{76}{75}a=(\frac{76}{75})^2a$; and at the end of the nth year $(\frac{76}{75})^na$.

Hence $(\frac{76}{75})^na=3a$, whence $n=\dfrac{\log 3}{\log 76-\log 75}=82.94$, or nearly 83 years.

Also solved by J. Scheffer, M. E. Graber, F. R. Honey, R. D. Carmichael, A. H. Holmes, G. W. Greenwood, L. E. Newcomb, and the Proposer.

233. Proposed by J. T. KEYES, Fogg High School, Nashville, Tenn.

At what time between 10 and 11 o'clock is the second hand of a clock one minute space nearer to the hour hand than it is to the minute hand?

I. Solution by J. SCHEFFER, Hagerstown, Md.

Let the time be x minutes past 10 o'clock. We assume that at the beginning of every minute the second hand points at 12 on the dial. The distance of the second hand at the required time is $60x-x=59x$; and that of the second hand from the hour hand is $60-60x-(10-\frac{1}{12}x)=50-60x+\frac{1}{12}x$.

$\therefore 59x=51-60x+\frac{1}{12}x$, whence $x=\frac{612}{1427}$ minutes$=25\frac{10}{14}\frac{45}{27}$ seconds.

II. Solution by G. W. GREENWOOD, M. A., Professor of Mathematics, McKendree College, Lebanon, Ill.

Let us measure the spaces clockwise from the minute hand to the second hand, and from the second hand to the hour hand. At n minutes and t seconds after ten, the number of minute spaces after the hour at which the hour hand, minute hand, and second hand stand are, respectively,

$$50+\frac{n}{12}+\frac{t}{720},\quad n+\frac{t}{60},\quad t.$$

$$\therefore 50+\frac{n}{12}+\frac{t}{720}-t+1=t-\left(n+\frac{t}{60}\right);\ i.\ e.,\ 36720+780n=1427t.$$

By putting $n=1, 2, \ldots\ldots, 59$, we get the corresponding values of t.

Also solved by Frederic R. Honey, A. H. Holmes, and L. E. Newcomb.

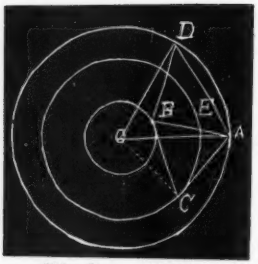
GEOMETRY.

257. Proposed by G. I. HOPKINS, A. M., Manchester, N. H.

Construire un triangle équilatéral sachant qu'el doit s'appuyer par ses trois sommets sur trois circonférences concentriques données. *Rouché et Comberousse.*

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Draw any radius OA of the largest circle, and on this construct the equilateral triangle OAD. From D as center with a radius OE of the middle circle draw an arc cutting the smallest circle at B, so that $DB=OE$, and draw BA, make $AC=AB$, and connect C with B, then ABC be the required triangle.



For, in the two triangles ABD and ACO, $AD=AO$, $BD=OC$, $AB=AC$.

$\therefore \triangle ABD=\triangle ACO$; $\therefore \angle DAB=\angle OAC$.

$\therefore \angle DAO=\angle BAC=60°$.

$\therefore \triangle ABC$ is an equilateral triangle having its vertices in the three concentric circumferences.

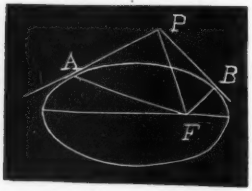
G. W. Greenwood called attention to the similarity of this problem to No. 250.

258. Proposed by B. F. FINKEL, A. M., Professor of Mathematics, Drury College, Springfield, Mo.

Prove that the tangents to an ellipse from any external point subtend equal angles at the focus, by means of the formula $\tan \phi = (m_1-m_2) / (1+m_1 m_2)$, where ϕ is the angle between the focal radius of either of the points of tangency and the line joining the focus and the external point, and m_1 and m_2 are the slopes of these two lines.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Designate the coördinates of the point of contact A by x_1, y_1, and those of the point of contact B, x_2, y_2, the equation of the ellipse being $a^2y^2+b^2x^2=a^2b^2$; then the equation of tangents PA and PB will be, respectively, $a^2yy_1+b^2xx_1 = a^2b^2$, and $a^2yy_2+b^2xx_2=a^2b^2$. By solving these as two simultaneous equations we find



$$x=\frac{a^2(y_2-y_1)}{x_1y_2-x_2y_1}, \quad y=-\frac{b^2(x_2-x_1)}{x_1y_2-x_2y_1},$$

which are the coördinates of the point P. Since the coördinates of focus F are $-ae$ and 0, we find the slope of PF

$$=-\frac{\frac{b^2(x_2-x_1)}{x_1y_2-x_2y_2}}{\frac{a^2(y_2-y_1)}{x_1y_2-x_2y_1}+ae}=-\frac{b^2(x_2-x_1)}{a[ay_2-ay_1+e(x_1y_2-x_2y_1)]},$$

and the slope of $AF=\frac{y_1}{x_1+ae}$.

$$\therefore \tan PFA =\left[-\frac{b^2(x_2-x_1)}{a[ay_2-ay_1+e(x_1y_2-x_2y_1)]}+\frac{y_1}{x_1+ae}\right]$$

$$\div\left[1-\frac{b^2y_1(x_2-x_1)}{a(x_1+ae)[ay_2-ay_1+e(x_1y_2-x_2y_1)]}\right]$$

$$= -\frac{(a+ex_1)[a^2y_1y_1+b^2x_1x_2-a^2b^2)]}{a^2(a+ex_1)[x_1y_2-x_2y_1-ae(y_1-y_2)\,]} = -\frac{a^2y_1y_2+b^2x_1x_2-a^2b^2}{a^2[x_1y_2-x_2y_1-ae(y_1-y_2)\,]}.$$

It is seen that this expression is symmetrical with reference to x_1 and x_2, y_1 and y_2 with the exception of the sign, but considering that by finding tanPFB the slope of BF comes first, it is at once seen that tanPFB is the same as tanPFA. The difficulty of this method lies in the complicated algebraic work, which is avoided by using polar coördinates.

Solution of 255 by Prof. William Hoover was received after the solution in last issue had gone to press. Also a solution of 256 was received from a contributor who failed to sign his name.

NOTE. Professor Matz sent in a solution of 254 in which he points out that the line $x-4a=0$ is both tangent and normal to the curve. But the solution is not general. Who can give a general solution?

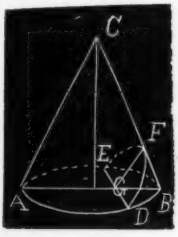
CALCULUS.

195. Proposed by CHRISTIAN HORNUNG, Heidelberg University, Tiffin, O.

Given a right cone of altitude h and radius r, to locate the plane parallel to its side which bisects the cone.

Solution by A. H. HOLMES, Brunswick, Maine, and J. SCHEFFER, Hagerstown, Md.

Let, in the right cone CAB, DEF represent a parabolic section. Put $BG=x$, $GE=y$, $FG=z$. The area of $DEF=\frac{4}{3}yz$; and consequently the volume of



$$BDEF=\tfrac{4}{3}\cdot\frac{h}{\sqrt{(r^2+h^2)}}\int_0^a yz\,dx.$$

and since $y^2=2rx-x^2$, and $z=\frac{x}{2r}\sqrt{(r^2+h^2)}$, we have for the volume, the integral

$$\tfrac{2}{3}\frac{h}{r}\int_0^x x\,dx\sqrt{(2rx-x^2)}=\tfrac{2}{3}\frac{h}{r}\left[\tfrac{1}{2}r^3\cos^{-1}\frac{r-x}{r}-\frac{3r^2+rx-2x^2}{6}\sqrt{(2rx-x^2)}\,\right].$$

To determine x for the condition that this volume is to be half the cone, we have the transcendental equation

$$2r^3\cos^{-1}\frac{r-x}{r}-\tfrac{2}{3}(3r^2+rx-2x^2)\sqrt{(2rx-x^2)}=r^3.$$

An approximate value of x is $x=1.3r$.

Also solved by R. D. Carmichael.

196. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

The shortest tangent intercepted by the axes of the ellipse to which the tangent is drawn, equals the sum of the semi-axes of the ellipse.

I. Solution by the PROPOSER.

Tangent$=y\sqrt{[1+(dx/dy)^2]}+x\sqrt{[1+(dy/dx)^2]}$, in which $y=(b/a)\times\sqrt{(a^2-x^2)}$ and $dy/dx=-bx/a\sqrt{(a^2-x^2)}$.

$$\therefore U=\frac{1}{a}\left[\frac{\sqrt{(a^2-x^2)}}{x}+\frac{x}{\sqrt{(a^2-x^2)}}\right]\sqrt{[a^4-(a^2-b^2)x^2]}$$

$$=a\sqrt{\left(\frac{a^4-(a^2-b^2)x^2}{x^2(a^2-x^2)}\right)}=\text{a minimum.}$$

$\therefore x=a^2/(a+b)$; and, consequently, the length of the required tangent becomes as stated in the problem.

II. Solution by G. W. GREENWOOD, M.A., Professor of Mathematics, McKendree College, Lebanon, Ill., and J. SCHEFFER, Hagerstown, Md.

Denote the length of the tangent by l, and its equation by $y=mx+\sqrt{(a^2m^2+b^2)}$.

$$\therefore l^2=(1+\frac{1}{m^2})(a^2m^2+b^2).$$

$$l^2m^2=a^2m^4+b^2+a^2m^2+b^2m^2=(am^2-b)^2+m^2(a+b)^2.$$

$$\therefore l^2=(a+b)^2+\left(\frac{am^2-b}{m}\right)^2.$$

Hence the minimum value of l is $a+b$.

Also solved by M. E. Graber, and W. L. Tryon.

197. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

$$\int_0^\infty \frac{\sin mx \cos nx}{x}dx.$$

Solution by G. W. GREENWOOD, M. A., Lebanon, Ill.; M. E. GRABER, Tiffin, Ohio, and the PROPOSER.

The required integral may be written

$$\tfrac{1}{2}\int_0^\infty \left[\frac{\sin(m+n)x}{x}+\frac{\sin(m-n)x}{x}\right]dx,$$

and it therefore reduces to problem No. 186, [January, 1905, page 22]. If $m+n$ and $m-n$ are both positive, the result is $\frac{1}{2}\pi$. If both negative, $-\frac{1}{2}\pi$. If of opposite sign, 0.

Also solved by S. A. Corey.

189. Proposed by M. E. GRABER, A. M., Heidelburg University, Tiffin, O.

Show that $e\int_1^\infty$, $e^2\int_2^\infty$, ... $e^n\int_n^\infty$, ... are integers divisible by $(p+1)!$, when the expression under the integral is $z^p\left[(z-1)\ldots\ldots(z-n)\right]^{p+1}e^{-z}\,dz$.

Solution by the PROPOSER.

$\int_0^\infty z^\rho[(z-1).....(z-n)]^{\rho+1}e^{-z}dz$ is divisible by $\rho!$ owing to the well known relation $\int_0^\infty z^\rho e^{-z}dz=\rho!$. Likewise, by substituting $z=z'+1$, $z=z'+2$, ..., $z=z'+n$, it can be seen that $e\int_1^\infty$, $e^2\int_2^\infty$ ..... $e\int_n^\infty$ ..... are divisible by $(\rho+1)!$.

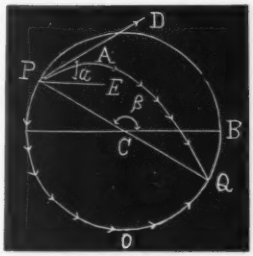
MECHANICS.

173. Proposed by J. F. LAWRENCE, A. B., Professor of Mathematics, Oklahoma Agricultural College. Stillwater, Oklahoma.

A squirrel is in a cylindrical cage and oscillates with it about its axis which is horizontal. At the instant when he is at the highest point of the oscillation, he leaps to the opposite extremity of the diameter and arrives there at the same instant as the point at which he left. Determine his leap completely.

Solution by B. F. FINKEL, A. M., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Let P be the highest point to which the squirrel ascends in his oscillations, the coördinates of this point being x_0, y_0; and Q the point opposite to which he jumps, the coördinates of this point being x_1, y_1, the origin of coördinates being taken at O.



Let the angle $BCP=\beta$, and let α be the angle between the horizon and the line of direction taken by the squirrel to reach the point Q.

Since the squirrel and cage are at rest when the squirrel reaches the highest point of his oscillations, and since, at that instant, the squirrel jumps to the point, Q, diametrically opposite, the cage is set in uniform motion in the direction POQ by the impulsive force of the squirrel's jump.

Let F_s be the impulsive force of the squirrel's jump; F_c, the effective impulsive force received by the cage; M_s, the mass of the squirrel; M_c, the mass of of the cage; V_s, the velocity of the squirrel; and V_c, the velocity of the cage. Then

$$F_c=F_s\sin(\beta-\alpha)\text{.....(1)},\quad F_c\tau=M_cV_c\text{.....(2)},\text{ and }F_s\tau=M_sV_s\text{.....(3)}.$$

The time for the point, P, to have the position, Q, is $t=\pi R/V_c$, and the time for the squirrel to reach the same point, by the parabolic path PAQ is $t=-\dfrac{2R\cos\beta}{\cos\alpha}$. Hence, $-\dfrac{2R\cos\beta}{V_s\cos\alpha}=\dfrac{\pi R}{V_c}$ ..... (4).

From (1), $F_c/F_s=\sin(\beta-\alpha)$, and from (2) and (3), $F_c/F_s=M_cV_c/M_sV_s$. Hence, $M_cV_c/M_sV_s=\sin(\beta-\alpha)$. Hence, $V_c=M_sV_s\sin(\beta-\theta)/M_c$.

Substituting this value of V_c in (4), and solving for α, we have

$$\tan\alpha=\tan\beta+\frac{\pi M_c}{2M_s\cos^2\beta}.$$

Knowing α, the other quantities may be found.

176. Proposed by A. H. HOLMES, Brunswick, Me.

A solid cube weighs 300 pounds. If a power is applied at an angle of 45° at an upper edge of the cube, how many foot-pounds will be required to overturn the cube?

Solution by CHRISTIAN HORNUNG, A. M., Heidelberg University, Tiffin, Ohio.

In order to overturn the cube it must be revolved on a lower edge until the center of mass is vertically over that edge, and this will require the lifting of the 300 pounds through a distance $a(\sqrt{2}-1)$, a being the edge of the cube, against gravity.

$\therefore$ the work done $=300a(\sqrt{2}-1)=124.26a$ foot-pounds. Hence, the *size* of the cube can not be left out of the calculation.

178. Proposed by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, O.

A weight W is drawn up a rough conical hill of height h and slope α. and the path cuts all the lines of greatest slope at the constant angle ϕ. Find the work done in attaining the summit.

[Problem 11, page 226, *Johnson's Theoretical Mechanics.*]

Solution by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

The weight W can be resolved in forces $W\cos\alpha$ perpendicular to the surface, $W\sin\alpha\cos\phi$ along its path, and $W\sin\alpha\sin\phi$ perpendicular to these directions. The force required to move the body is therefore $\mu W\cos\alpha+W\sin\alpha\cos\phi$.

The path is the equiangular spiral $s=r\sec\phi$, and hence its length from the foot of the hill is $n\operatorname{cosec}\alpha\sec\phi$.

The work done is then

$$n\operatorname{cosec}\alpha\sec\phi(\mu W\cos\alpha+W\sin\alpha\cos\phi);\ i.\ e.,\ Wh(\mu\cot\alpha\sec\phi+1).$$

179. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

If the *velocity* of a body moving under an acceleration tending to the center *varies* as the radius of curvature, the body will describe a cycloid.

Solution by the PROPOSER.

Assume $x=\text{ver-sin}^{-1}(y)-\sqrt{(2ry-y^2)}$ to represent the orbit; then $R,=v$, $=2\sqrt{(2ry)}$, which fulfills the conditions of the problem.

DIOPHANTINE ANALYSIS.

126. Proposed by R. A. THOMPSON, M. A., C. E., Engineer Railroad Commission of Texas.

Eight persons wish to play a series of games of progressive duplicate whist. In one evening, 12 boards are played, 4 boards (and return) by one couple against each of the other three couples, the same partners being retained throughout one evening. How many evenings will be required to complete the series, and what is the order of play, it being required that each player shall play with every other player as partner, and that each couple shall play once and but once against every other couple.

Solution by A. H. HOLMES, Brunswick, Maine.

The order of play would be as follows, the first eight letters representing the players:

A–B vs. C–D and E–F vs. G–H
A–B vs. E–F and C–D vs. G–H
A–B vs. G–H and C–D vs. E–F
} first evening.

The arrangement of couples for the second evening would be: A–C vs. B–D and E–G vs. F–H, alternating as first evening.

Third evening: A–D vs. F–G and B–C vs. E–H.
Fourth evening: A–E vs. B–F and C–G vs. D–H.
Fifth evening: A–F vs. B–G and C–H vs. D–E.
Sixth evening: A–G vs. B–H and C–E vs. D–F.
Seventh evening: A–H vs. C–F and B–E vs. D–G.

It would therefore take seven evenings to *make* the series, or six evenings to *complete* the series.

AVERAGE AND PROBABILITY.

159. Proposed by J. E. SANDERS, Hackney, Ohio.

A box contains n tickets numbered from 1 to n. How many draws, on the average, will it take to draw all the numbers, each ticket being replaced before drawing again? What is the numerical result for $n=2$ and $n=6$?

REMARKS. Mr. Corey and the Proposer insist that the published solution of this problem in the May Number is incorrect. Mr. Sanders gets for $n=2$, $p=\frac{1.2}{2}+\frac{1.3}{2^2}+\frac{1.4}{2^3}+\frac{1.5}{2^4}+\ldots\ldots=3+$; and for $n=3$, $p=\frac{2.3}{3^2}+\frac{2.4}{3^3}+\frac{14.5}{3^4}+\frac{10.6}{3^4}+\frac{62.7}{3^8}+\ldots\ldots$, the sum of the first ten terms of which is 5.1. From a large number of actual trials, he obtained the following results: $n=2$, $p=2.9$; $n=3$, $p=5.5$; $n=4$, $p=8.5$; $n=5$, $p=12.31$; $n=6$, $p=15.67$; $n=8$, $p=22.66$; $n=12$, $p=41$,—values which are (he observes) approximately equal to $\sqrt{n^3}$.

As stated in the **May** Number, the published solution conforms to the solution of Problem IX, page 52, of Meyer's *Wahrscheinlichkeitsrechnung*, which problem I take to be similar to the one under consideration. The formula used by Professor Landis in his solution is the same as that obtained by Meyer, a formula which is only approximate, but holds with great exactness if n is large. The correct result is to be obtained from the equation

$$\tfrac{1}{2}=1-n\left(\frac{n-1}{n}\right)^i+\frac{n(n-1)}{2!}\left(\frac{n-2}{n}\right)^i-\frac{n(n-1)(n-2)}{3!}\left(\frac{n-3}{n}\right)^i$$

$$+\frac{n(n-1)(n-2)(n-3)}{4!}\left(\frac{n-4}{4}\right)^i+, \text{ etc.},$$

where n is the number of numbers and i the number of drawings. This is Meyer's answer to his Problem VIII, which reads: Eine Lotterie besteht aus n Nummern, in jeder Ziehung wird eine davon gezogen. Es soll die Wahrscheinlichkeit π gefunden werden, dass in i Ziehungen alle nummern erschienen sind.

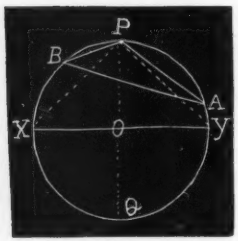
Professor Matz insists that the solution of Problem 158 in the May Number of the MONTHLY is also incorrect, his contention being that to take an arc for every point on AC and BD would not be taking them uniformly on AB. In reference to this contention, we must again call attention to Dr. Moore's *Note on Mean Value*, page 303, Vol. II of MONTHLY. The problem as stated is indefinite, and thus one may choose any law of distribution he wishes. Accordingly, the published solution is correct according to the law of distribution chosen. Other laws of distribution may be chosen, giving other results. ED. F.

161. Proposed by F. P. MATZ. Sc. D., Ph. D., Reading, Pa.

A triangle is inscribed at random in a circle; (a) what is the chance the triangle is *oblique;* and (b) what is the chance the triangle is *less in area* than $\frac{1}{4}\pi r^2$?

Solution by the PROPOSER.

(I) Put $OP=r$, $\angle APO=\theta$, $\angle BPO=\phi$; then $\triangle APB=2r^2\cos\theta\cos\phi\times\sin(\theta+\phi)$. In order that $\triangle APB$ may be *obtuse*-angled at P, the point A is restricted to the arc PAX, and the point B to the arc PBY. The required chance, therefore, becomes



$$C_1=\frac{2r^2\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}\int_{\frac{1}{4}\pi}^{\phi}\cos\theta\cos\phi\sin(\theta+\phi)d\phi d\theta}{2r^2\int_0^{\frac{1}{2}\pi}\int_0^{\phi}\cos\theta\cos\phi\sin(\theta+\phi)d\phi d\theta}$$

$$=\frac{\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}[2\sin^2\phi\cos^2\phi-\frac{1}{2}\cos^2\phi+(\phi-\frac{1}{4}\pi-\frac{1}{2})\sin\phi\cos\phi]d\phi}{\int_0^{\frac{1}{2}\pi}[2\sin^2\phi\cos^2\phi+\phi\sin\phi\cos\phi]d\phi}$$

$$= \left(\frac{\pi}{16} - \frac{1}{8}\right) / \frac{\pi}{4} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{\pi}\right).$$

(II) In this case the *superior* limit of ϕ in the numerator of C_1 is the value of ϕ derived from the equation $\sin\phi \cos^3\phi = \frac{1}{16}\pi$; and the *inferior* limit of the same variable is zero.

The required chance C_2 can, therefore, be found approximately; but is not of sufficient interest to warrant the labor required to find it.

MISCELLANEOUS.

147. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

If P be a point within the scalene triangle, such that $\angle PAB = \angle PBC = \angle PCA = \phi$, then $\cot\phi = \cot A + \cot B + \cot C$ ..... (1), and $\operatorname{cosec}^2\phi = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C$ ..... (2).

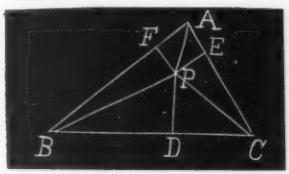
I. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Let $\angle PAB = \angle PCA = \angle PBC = \phi$. Then $\angle APB = \pi - (\phi + B - \phi) = \pi - B$. Draw PD, PE, PF perpendicular to BC, CA, AB, respectively.

$$PD = PB\sin\phi = \frac{AB \sin PAB}{\sin APB} \cdot \sin\phi = \frac{c\sin^2\phi}{\sin B}$$

$$= \frac{2Rc}{b}\sin^2\phi. \quad \text{So } PE = 2R\frac{a}{b}\sin^2\phi,$$

$$PF = 2R\frac{b}{a}\sin^2\phi.$$



$$\frac{\sin(A-\phi)}{\sin\phi} = \frac{PE}{PF} = \frac{a^2}{bc} = \frac{\sin A \sin(B+C)}{\sin B \sin C}.$$

$\therefore \cot\phi - \cot A = \cot B + \cot C$, or $\cot\phi = \Sigma\cot A$.

Also $\cot^2\phi = \Sigma\cot^2 A + 2\Sigma\cot B \cot C$, $\operatorname{cosec}^2\phi - 1 = \Sigma\operatorname{cosec}^2 A - 3 + 2$;

i. e., $\operatorname{cosec}^2\phi = \Sigma\operatorname{cosec}^2 A$.

II. Solution by the PROPOSER.

Let $PA = m$, $PB = n$, $PC = p$. $\sin(\pi - B) : \sin(B - \phi) = c : m$.

$\therefore \cot\phi - \cot B = (m/c \sin\phi) = 2\cot B$ ..... (1).

Also, $\cot\phi - \cot C = (p/a \sin\phi) = 2\cot C$ ..... (2);

and $\cot\phi - \cot A = (n/b \sin\phi) = 2\cot A$ ..... (3).

Adding, and dividing by (3), we have $\cot\phi = \cot A + \cot B + \cot C$ ..... (*A*).

Squaring (*A*), and transforming into cosecants, we have

$$\operatorname{cosec}^2\phi = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C.$$

Also solved by M. E. Graber, J. Scheffer, and A. H. Holmes.

GROUP THEORY.

7. Proposed by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

Which linear substitution will transform $x_1x_2+x_3x_4+x_5x_6=0$ into $y_1^2+y_2^2+y_3^2-y_4^2-y_5y_6=0$?

Remark by the PROPOSER.

One substitution having the desired property is

$$\left\{\begin{matrix} x_1, & x_2, & x_3, & x_4, & x_5, & x_6, \\ y_1+iy_2, & y_1-iy_2, & y_3+y_4, & y_3-y_4, & -y_5, & y_6, \end{matrix}\right\}^*$$

where $i=\sqrt{-1}$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

237. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Solve $x^2+y+z=12$.....(1); $x+y^2+z=8$.....(2); $x+y+z^2=6$.....(3).

238. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that $\frac{1}{1+n}+\frac{1}{3(n+3)}+\frac{1}{5(n+5)}+\text{.....}=\frac{1}{2}\left[\frac{1}{(n-1)}+\frac{1}{3(n-3)}+\frac{1}{5(n-5)}+\text{....}+\frac{1}{l(n-l)}\right]$, n being an even positive integer and $l=n-1$.

239. Proposed by J. J. KEYES, Fogg High School, Nashville, Tenn.

Solve $\sqrt[4]{(41+x)}+\sqrt[4]{(41-x)}=4$.

GEOMETRY.

260. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Perpendiculars to the radius vector are drawn through points on $r=a+b\cos n\theta$. Find the radius of curvature of their envelope at a point at a given distance from the origin.

261. Proposed by R. D. CARMICHAEL, Hartselle, Alabama.

Given three non-intersecting circles; to draw eight tangent circles, each tangent to all three of the given circles.

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*More generally, one set of substitutions fulfilling the required conditions, is

$$\left\{\begin{matrix} x_1, & x_2, & x_3, & x_4, & x_5, & x_6, \\ ay_1\pm aiy_2, & ay_1\mp aiy_2, & ay_3\pm ay_4, & ay_3\pm ay_4, & \pm ay_5, & \mp ay_6, \end{matrix}\right\}$$

where a is not equal 0. ED. E.

262. Proposed by NELSON L. RORAY, Utica, New York.

In a regular pentagon, show that $\frac{\text{diagonal}}{\text{side}} = \frac{2 \text{ apothem}}{\text{radius}} = \frac{1}{2}\cdot\frac{5+\sqrt{5}}{\sqrt{5}}$.

CALCULUS.

199. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

If the perimeter of an ellipse varies uniformly at the rate of $\frac{1}{4}$ inch per unit of time, at what rate is the eccentricity varying the instant the perimeter becomes 60 inches and the major axis 25 inches?

200. Proposed by R. D. CARMICHAEL, Hartselle, Alabama.

Find the equation of a curve so that the area bounded by the curve, the axis of x, and any ordinate y, is equal to $y-x$, x being the *corresponding* abscissa.

201. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Solve $\int\int^{dy/dx} \frac{dw}{1+w^2} = 0$.

MECHANICS.

180. Proposed by EDWIN L. RICH, Lehigh University, South Bethlehem, Pa.

If a body is projected into the air, and the resistance of the air varies as the square of the velocity; required the equation of the curve. [From De Volson Wood's *Analytical Mechanics*, problem 10, p. 179.]

181. Proposed by F. ANDEREGG, Professor of Mathematics, Oberlin College, Oberlin, Ohio.

A triangle AOB, of which the sides, OA, AB, and the angle at O are a, b, and α, revolves uniformly about O, so that OA makes the angle nt with the axis of x, and carries a circle of which AB is the diameter. Prove that a point moving in the circumference of the carried circle with twice the angular velocity of the triangle will describe an ellipse whose axes are

$$\sqrt{(a^2+b^2+2ab\cos\alpha)} \pm \sqrt{(a^2+b^2-2ab\cos\alpha)}.$$

182. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

I have a tank, the lower part of which is a hemisphere 22 feet in diameter. The rest is a cylinder 22 feet in diameter, and altitude 28 feet. This tank is connected with the earth by a vertical stand-pipe 10 inches in diameter, 130 feet long, extending 2 feet into the tank. The tank is filled by a $2\frac{1}{2}$ inch pipe 65 feet long, having one right-angled elbow delivering the water into the bottom of the stand-pipe from a steam pump under 96 pounds gauge pressure. How long will it take to fill the pipe?

DIOPHANTINE ANALYSIS.

127. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Can there be determined three cube numbers whose sum is the product of two squares?

AVERAGE AND PROBABILITY.

165. Proposed by HENRY HEATON, Atlantic, Iowa.

What is the average length of all straight lines that can be drawn within a given square parallel to one of the diagonals?

166. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Find the average area intercepted by two non-intersecting chords drawn at random in a given circle.

167. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

A line l is divided into n segments by $n-1$ points taken at random on it; find the mean value of the product of p of the segments, the p segments being taken at random and p being less than n.

GROUP THEORY.

8. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

In a chess tournament between eight players, there are seven rounds, the eight players being paired in each round, every pair to be matched once and but once in the tournament. List the possible programs different except as to notation, *i. e.*, not transformable into each other by a substitution on eight letters. Give the number of conjugate programs of each representative retained.

MISCELLANEOUS.

149. Proposed by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

Given $\sin^{-1}u+\sin^{-1}\frac{1}{2}u=\frac{1}{4}\pi$, to find u.

NOTES.

Dr. Saul Epsteen has been elected to the chair of Applied Mathematics in the University of Colorado, made vacant by the resignation of Dr. Arnold Emch who has accepted the professorship of Mathematics in his alma mater in Switzerland. F.

The April Number of the *Educational Times*, London, England, announced the death of Rev. Allen Whitworth, Prebendary of St. Paul's Cathedral. Mr. Whitworth was among the earliest contributors to the mathematical columns of the *Educational Times*, his name occurring in the list of contributors to the first volume of the Reprint. He was a recognized authority in pure probability, and his little book, "Choice and Chance," together with his "DCC Exercises" afford the most reliable source from which to gain information of the essential principles underlying the subject. F.

With this issue of the MONTHLY Dr. Saul Epsteen, who has so ably edited the Department of Problems and Solutions in the MONTHLY from October, 1903, to January, 1905, and who very kindly consented to assist in editing that Department since January, 1905, will retire from the editorial staff. It will be remembered that Dr. Epsteen assumed this work in order to relieve us from the care and responsibility while attending the University of Pennsylvania. We take this opportunity to express our keen appreciation of his services and to thank him for the interest, efficiency, and care he has taken in the work. F.

BOOKS AND PERIODICALS.

Supplementary Algebra. By R. L. Short. Small 8vo, Paper Cover. 45 pages. Boston: D. C. Heath & Co.

This monograph was written with a view to satisfy the demands coming from teachers for supplementary work in algebra. The graph is used to represent functions, and many of the illustrations have been taken from the calculus and mechanics. A number of excellent short methods in reduction and the solution of equations are introduced. F.

Examples in Algebra. Eight Thousand Exercises and Problems Carefully Graded from the Easiest to the Most Difficult. By Charles M. Clay, Head Master of Roxbury High School, Boston, Mass. 8vo, Cloth, viii+372 pages. Price, 90 cents. New York: The Macmillan Co.

The author sounds the key-note of successful algebra teaching, when he asserts that the first need of beginners in algebra is drill, drill, drill until the fundamental processes become as familiar to them as in arithmetic. He believes that too much time put in upon problems teaches the student to think rather than to act, and too much upon examples to act rather than to think, and so a judicious mixture of both is the only wise course. To this end these examples and problems are supplied. They are carefully graded and are well chosen. F.

Essentials of Algebra. Complete Course for Secondary Schools. By John C. Stone, A. M., Michigan State Normal School, and James F. Mills, A. M., The Shortridge High School, Indianapolis, Ind. 8vo, Half Leather, xii+412 pages. Boston, New York, Chicago: Benjamin F. Sanborn & Co.

Some of the special features of this book are, that the pupil is taught from the beginning how to check his work; he is required to see that every step in the solution of an equation is the application of a fundamental principle, "transposing" and "clearing of fractions" being used only after he thoroughly knows the underlying processes; the exercises, which are numerous and well graded; and the graph. The book is one which we believe will bear searching criticism. F.

Elementary Algebra. By Walter R. Marsh, Head Master Pingry School, Elizabeth, N. J. 8vo, Half Leather, vi+395 pages. New York: Charles Scribner's Sons.

This text follows the requirements of the College Entrance Examination Board as to subject matter, emphasis being placed upon those principles which are to be the tools of subsequent mathematical study. Many of the problems and illustrations are taken from physics and other practical sources. F.

The American Journal of Mathematics. Edited by Frank Morley and Simon Newcomb, and others. Published Quarterly under the auspices of the Johns Hopkins University.

The April Number contains the following articles: On a Class of Differential Equations, by A. Chessin; Surfaces with the Same Spherical Representation of their Lines of Curvature as Pseudo-Spherical Surfaces, by L. P. Eisenhart; On the Forms of Sextic Scrolls having no Rectilinear Directrix, by V. Snyder; Determination of the Ternary Modular Group, by L. E. Dickson. F.

The Annals of Mathematics. Published Quarterly under the auspices of Harvard University.

The April Number contains the following articles: Groups of the Fundamental Operations of Arithmetic, by G. A. Miller; Linear Differential Equations with Discontinuous Coefficients, by M. Bocher; Some Physical Solutions of the Equation of the nth Degree, by R. E. Moritz; The Ballistic Problem, by F. Gilman; A Theorem Concerning Uniform Convergence, by G. D. Birkhoff. F.

ERRATA.

Page 108, line 5, for "$(1.2.3.....a)^4 \equiv 0$" read $(1.2.3.....a)^4 - 1 \equiv 0$.
Page 136, line 9 from bottom, for "$x_1y_2 - x_2y_2$" read $x_1y_2 - x_2y_1$.

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GROUPS CONTAINING THE LARGEST POSSIBLE NUMBER OF OPERATORS OF ORDER TWO.

By G. A. MILLER.

The abelian group of order 2^m and of type (1, 1, 1, ...) contains 2^m-1 operators of order two. This group will not be considered in what follows. Let G represent any other group whose order (g) is divisible by the prime number p. We proceed to prove that the number of the operators of G whose orders exceed two is at least $\frac{p-1}{2p}g$ and to determine all the possible groups in which the number of these operators is exactly $\frac{p-1}{2p}g$. The considerations are extremely simple but the results seem of sufficient importance to be given in an explicit form.

If g is divisible by 2 but not by 4 it is well known that G contains a subgroup (H) of half its order which is composed of its operators of odd order in addition to the identity. In such a group the number of operators of order 2 cannot exceed $g/2$, and whenever the number of operators of order two contained in G is exactly $g/2$ then the order of G is twice an odd number and the subgroup H is abelian. These known facts are stated here merely for the sake of completeness. They are not essential in what follows.

If g is divisible by 4 it is clearly possible to construct a G such that more than half its operators are of order 2. Such a group is obtained by extending the cyclic group of order $g/2$ by the operators of order two which transform each operator of this cyclic group into its inverse. The group obtained in this man-